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Partial derivative approach for option pricing in a simple stochastic volatility model

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Abstract. We study a market model in which the volatility of the stock may jump at a random time from a fixed value to another fixed value. This model has already been introduced in the literature. We present a new approach to the problem, based on partial differential equations, which gives a different perspective to the issue. Within our framework we can easily consider several forms for the market price of volatility risk, and interpret their financial meaning. We thus recover solutions previously mentioned in the literature as well as obtaining new ones.

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1 Introduction

The problem of pricing financial derivatives was already present in the aim of early works in Mathematical Finance. In 1900, Bachelier [1] proposed arithmetic Brownian motion for the dynamical evolution of stock prices as a first step towards obtaining a price for an option. Nevertheless, interest in this problem has increased remarkably in the past thirty years, after the publication of the works of Black and Scholes [2], and Merton [3]. The Black-Scholes model has been broadly used by practitioners since then, mainly due to its mathematical simplicity. It is well established, however, that this model fails to explain some statistical features shown in real markets. In particular, there is solid evidence pointing to the necessity of relaxing the assumption, present in the Black-Scholes model, that a constant volatility parameter drives the stock price. One of most commonly used tests is based on a conceptually simple principle. Since the Black-Scholes price is a monotonous function on its arguments, the formula can be inverted in order to compute the *implied volatility*, i.e. the volatility that will reproduce the actual market conditions. The usual result is that the implied volatility is not constant, but a U-shaped function of the moneyness, whose minimum is at moneyness near to one —i.e. when the current price of the underlying is close to the strike. This departure from the Black-Scholes model is known as the *smile effect*, and it is well documented in the literature [4].

Many models have been developed with the purpose of avoiding this restrictive feature. We will mention only a few of them here. Merton himself [5] proposed a model in which volatility was a deterministic function of time. Cox and Ross [6] presented some alternative proposals that can be thought of as models in which the volatility is stock-dependent. These and other similar contributions lead to a framework in which all the option risk comes from the fluctuations in the price of the underlying. In practical situations, however, it seems that this description is not sophisticated enough for explaining the actual changes in the level of volatility. Some authors have then suggested that the evolution of the volatility is driven by its own stochastic equation. Among these models of *stochastic volatility* we find works that are historically noteworthy. Hull and White [7] proposed a model where the squared volatility also follows a log-normal diffusion equation, regardless of the stock price. Wiggins [8] extended this idea and considered that the underlying and the volatility constitute a two-dimensional system of correlated log-normal random processes. Scott [9], and Stein and Stein [10] in particular, assumed that instantaneous volatility follows a random mean-reverting process: an independent arithmetic Ornstein-Uhlenbeck process. Masoliver and Perelló [11] relaxed this assumption, and introduced correlation into the two-dimensional Wiener process. Heston [12] turned the arithmetic model into a square-root correlated process.

The common feature of all these seminal papers is that they model the stochastic behaviour of volatility as a diffusion process. Naik [13] developed a model in which the volatility can take only two known values, and the market switches back and forth between them, in a random way. This set-up can be used to model a market with high and

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low volatility periods. Herzel [14] studied a simplified version of this problem, in which the volatility can at most jump once. This is a suitable model for encoding a market that may undergo a severe change in volatility only if some forthcoming event takes place. Since options have a limited lifetime, this does not seem to be a very restrictive limitation. Herzel formally solved the problem of pricing the options using probability arguments, and showed that his model can account for the smile effect.

We present here a different approach for obtaining fair option prices under Herzel's assumptions. We will employ a technique used extensively both in research papers (e.g. in [12]) and reference books (e.g. in [15]) on this topic: we determine the partial differential equation that the option price must fulfil, according to Itô's formula, and solve it with the appropriate constraints. This scheme eases the task of considering different forms for the market price of volatility risk, and leads to a simple way of interpreting the financial meaning of each.

The paper is structured as follows: in Section 2 we present the general market model and specify the differential equations that govern the traded securities. In Section 3 we study the way of obtaining a complete market. In Section 4 we explore the consequences of demanding that the market admits no arbitrage. In Section 5 we present explicit solutions when the market price of the volatility risk is uniform in time, for two typical contract specifications. In Section 6 we consider more complex scenarios in which the volatility premium is not constant but depends on different variables. Section 7 contains actual numerical examples of several solutions and goes into further depth in the financial interpretation of the results. The conclusions are drawn in Section 8. The paper ends with Appendix A, where we detail an alternative approach that gives more financial insights to one of the new solutions we have introduced.

2 The market model

Let us begin with the general description of our set-up. We will assume that in our market there is at least a non-deterministic traded stock, S. The evolution of the price of this stock, from S_0 at $t = t_0$, is governed by the following differential equation[1]:

$$\frac{dS}{S} = \mu dt + \sigma dW, \qquad (1)$$

where $W(t - t_0)$ is a one dimensional Brownian motion, with zero mean and variance equal to $t - t_0$, μ is a constant parameter, and σ, the volatility, is a stochastic quantity. The model assumes that the volatility initially has a given value σ_a, and that at most it may change to a different value σ_b at instant $\tau > t_0$:

$$\sigma(t;\tau) = \sigma_a \mathbf{1}_{t<\tau} + \sigma_b \mathbf{1}_{t\geq\tau} = \sigma_a + (\sigma_b - \sigma_a)\mathbf{1}_{t\geq\tau}, \qquad (2)$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function, which assigns the value 1 to a true statement, and the value 0 to a false statement. The time τ in which such transition occurs is random and we will assume that it follows an exponential law:

$$P(t_0 < \tau \leq t) = 1 - e^{-\lambda(t-t_0)}.$$

Note that with the previous definition, λ is merely the inverse of the mean transition time, $E[\tau - t_0] = \lambda^{-1}$.

We also assume that we will be capable of concluding whether the transition has taken place or not. This assumption does not imply that we can directly measure the value of σ, but that there is a way to determine if $t \geq \tau$. This can be easily understood from the point of view of a practitioner. Let us suppose, for instance, that we are expecting a relevant financial announcement to be made. We do not know for sure when this will happen, but we believe that the new information will affect the level of volatility in our market. Even though we may not perform an instantaneous measure of the volatility in order to check the actual effect of the news, we can know if they are simply published. We will return to this issue later.

For the underlying stock S, we will define a new traded asset: the option C. The price of this option will depend explicitly on the moment t_0 in which we will decide to evaluate it, on the current stock price S_0 and on the level of volatility σ_0, but also on a set of particular parameters which we will label with a single symbol, κ. These parameters are the contract specifications that will characterize the option, including the maturity time or the striking price. This framework covers the European put and call options, e.g. the *vanilla* options or the Binary options, and the American options as well, but does not include more exotic derivatives, such as the Asian options or the lookback options.

The differential of the option price $C = C(t, S, \sigma; \kappa)$ has, according to Itô's formula, the following expression:

$$dC = \partial_t C dt + \partial_S C dS + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} C dt + \frac{\Delta C}{\sigma_b - \sigma_a} d\sigma, \qquad (3)$$

where

$$\Delta C \equiv C(t, S, \sigma_b; \kappa) - C(t, S, \sigma_a; \kappa).$$

The last term in equation (3) condenses innovation with respect to the classical Black-Scholes expression, and represents the contribution of the randomness in the volatility to the dynamics of the option price. Note that this extra term is a product of the finite difference version of the derivative of C with respect to σ, and $d\sigma$. In order to obtain an alternative expression for this object, we will simply differentiate equation (2):

$$d\sigma = (\sigma_b - \sigma_a) d\mathbf{1}_{t\geq\tau}. \qquad (4)$$

The differential of an indicator with a random variable in its argument may seem a bizarre object. However, it is mathematically well defined, as we will shortly show. We can decompose this differential in two terms:

$$d\mathbf{1}_{t\geq\tau} = \lambda \mathbf{1}_{t<\tau} dt - \lambda dG. \qquad (5)$$

[1] Throughout our exposition we will not specify the explicit dependence of the involved magnitudes, unless this may lead to confusion.

The first term is regular, and the second involves G,

$$G = t\mathbf{1}_{t<\tau} + \left(\tau - \frac{1}{\lambda}\right)\mathbf{1}_{t\geq\tau},$$

which is proven to be a right continuous with left limits martingale. Nevertheless, in order to make the notation easier, we will keep the differential of the indicator in its early form, and use the referred decomposition only when it can clarify the problem. However, we must stress that $d\mathbf{1}_{t\geq\tau}$ is a stochastic magnitude, independent of dW. Since $d\sigma$ does not directly contribute to the variation of the stock price dS, we can foresee that there is a source of risk that cannot be explained in terms of the random evolution of the underlying asset. However, we will deal with this issue in the next section.

Before doing so, we would like to point out that there is also a third kind of security traded in the market, a free-risk monetary asset B, which satisfies:

$$dB = rBdt. \tag{6}$$

This security will provide a secure resort where to keep the benefits of an effective investment strategy, but also allow us to borrow money when we need it. Therefore, it makes possible both the *self-financing strategy*, which allows closed portfolios, and the *net-zero investment*, the composition of a portfolio with no net value.

3 Completeness of the market

Let us face the problem of the completeness of the market. It is well-known that the market will be complete if we can construct the so-called *replicating portfolio* for every security, i.e. a portfolio that mimics the behaviour of the asset. We have argued in the previous section that not all the influence of σ on the price of the option can be explained through S. We therefore need another security that can account for this component of the global risk. Instead of introducing a new traded asset depending only on $d\sigma$, with no clear financial interpretation, we have decided to use a *secondary* option $D(t,S,\sigma;\kappa')$: a derivative of the same nature of $C(t,S,\sigma;\kappa)$, but with a different set of contract specifications. This add-on completes the market if we are allowed to borrow money at a fixed interest rate whenever we need it, or to buy zero-coupon bonds in the event that we obtain a surplus of cash. We can thus write down C as a combination of δ shares S, ϕ units of the riskless security B, and ψ secondary options D:

$$C = \delta S + \phi B + \psi D.$$

The variation in the value of both portfolios fulfills

$$dC = \delta dS + \phi dB + \psi dD,$$

where we have taken into account two capital facts. On one hand δ, ϕ and ψ are predictable functions of S and D, e.g. $d\delta$, $d\phi$ and $d\psi$ do not depend on the new random information in dW and $d\sigma$. On the other hand, we adopt a *self-financing strategy*, in which there is no net cash flow entering or leaving the replicating portfolio [16]:

$$Sd\delta + Bd\phi + Dd\psi = 0.$$

We will replace dC with the expression in (3), and we will take into account the properties shown in (4) and (6), to finally obtain:

$$\partial_t C dt + \partial_S C dS + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} C dt + \Delta C d\mathbf{1}_{t\geq\tau} = \delta dS + r\phi B dt + \psi dD. \tag{7}$$

We can proceed with dD in an analogous way,

$$dD = \partial_t D dt + \partial_S D dS + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} D dt + \Delta D d\mathbf{1}_{t\geq\tau}, \tag{8}$$

where the natural definition of ΔD,

$$\Delta D = D(t,S,\sigma_b;\kappa') - D(t,S,\sigma_a;\kappa'),$$

has been used. In order to recover a deterministic partial differential equation we must guarantee that all the terms containing the stochastic magnitudes dS and $d\mathbf{1}_{t\geq\tau}$ mutually cancel out. Thus we must demand that

$$\partial_S C = \delta + \psi \partial_S D,$$

condition named *delta hedging*, and also that

$$\Delta C = \psi \Delta D,$$

which is usually referred as *vega hedging*, or sometimes as *psi hedging* [17].

The previous hedging conditions reduce equation (7) to

$$\partial_t C dt + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} C dt = r\phi B dt + \frac{\Delta C}{\Delta D}\left(\partial_t D dt + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} D dt\right), \tag{9}$$

an expression that still involves B, which is not an inner variable of the option prices C and D. This problem can be fixed using the definition of the portfolio and the *psi hedging* together,

$$\phi B = C - \delta S - \psi D = C - \left(\partial_S C - \frac{\Delta C}{\Delta D}\partial_S D\right) S - \frac{\Delta C}{\Delta D} D.$$

The replacement of ϕB in equation (9) thus leads to

$$\partial_t C + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} C - rC + rS\partial_S C = \frac{\Delta C}{\Delta D}\left(\partial_t D + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} D - rD + rS\partial_S D\right).$$

This formula implies the existence of an arbitrary function $\chi = \chi(t,S,\sigma)$, which uncouples the problem of finding C and D:

$$\chi(t,S,\sigma) = \frac{1}{\Delta C}\left(\partial_t C + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} C - rC + rS\partial_S C\right). \tag{10}$$

Obviously the same formula is valid for the secondary option, merely by replacing C with D. This fact proves that the option D completes the market indeed [17,18]. Note that if we set $\chi(t,S,\sigma) = 0$ we recover the classical Black-Scholes equation.

4 Arbitrage-free scenario

We have to state some criterion before we can choose a valid candidate for $\chi(t,S,\sigma)$. We will first determine the meaning of this arbitrary function. Consider Y, a portfolio which involves shares, bonds, one primary option, and secondary options:

$$Y = C + \bar{\delta} S + \bar{\phi} B + \bar{\psi} D. \tag{11}$$

We will choose the relative amount of each security in such a way that at the beginning, the portfolio has no net value, i.e. $Y = 0$. Moreover, we will also require the changes in the value of the portfolio not to come from a cash flow. These assumptions will ease the forthcoming discussion because, in demanding that, we have removed both riskless and externally induced growth of the portfolio. Imagine now that we can show that, under these conditions, the future evolution in the valuation of the portfolio never becomes negative. We would have designed a financial instrument for making money, with safety and no initial investment required. In other words, this would be proof that our market presents arbitrage opportunities. Obviously, a reciprocal scenario in which a change in the value of the portfolio necessarily implies a depreciation also leads to arbitrage. We may just build the opposite portfolio, $\bar{Y} = -Y$, and obtain guaranteed profits without any exposure of capital. The presence of arbitrage in other portfolios will lead back to the same constraints on Y. We must therefore analyse dY and prevent it from having a definite sign.

The predictable nature of $\bar{\delta}$, $\bar{\phi}$, and $\bar{\psi}$ makes dY take the following form:

$$dY = dC + \bar{\delta} dS + \bar{\phi} dB + \bar{\psi} dD. \tag{12}$$

First we can combine equations (6) and (11),

$$\bar{\phi} dB = r\bar{\phi} B dt = r\left(Y - C - \bar{\delta} S - \bar{\psi} D\right),$$

to cancel out the bond term in (12):

$$dY = dC + \bar{\delta} dS - r\left(C + \bar{\delta} S + \bar{\psi} D\right) dt + \bar{\psi} dD.$$

Then we can use equations (3), (4), (8), and (10) in order to obtain:

$$dY = (\Delta C + \bar{\psi}\Delta D)\left(\chi dt + d\mathbf{1}_{t\geq\tau}\right) + (\bar{\delta} + \partial_S C + \bar{\psi}\partial_S D)(dS - rSdt).$$

Finally, we will remove all the dependence in dS, just setting $\bar{\delta} = -\partial_S C - \bar{\psi}\partial_S C$,

$$dY = (\Delta C + \bar{\psi}\Delta D)\left(\chi dt + d\mathbf{1}_{t\geq\tau}\right). \tag{13}$$

The reason for this last step is quite simple. Note that $dS - rSdt = (\mu - r)Sdt + \sigma S dW$ can have any real value. That is, it cannot be a source of arbitrage possibilities, adding nothing but uncertainty in the behaviour of the null portfolio. If we are looking for some sort of predictability in the sign of dY, we must therefore make that term disappear.

An initial analysis of equation (13) quickly reveals that the sign of dY depends in turn on the behaviour of both factors: $(\Delta C + \bar{\psi}\Delta D)$ and $(\chi dt + d\mathbf{1}_{t\geq\tau})$. Let us check the first. The sign of ΔC and ΔD is predictable, because option prices have a monotonous response to changes in the volatility. The value of $\bar{\psi}$ can be chosen at our convenience in this portfolio, and as a result, unless we set it equal to $\bar{\psi} = -\Delta C/\Delta D$, the first term will have a definite sign. We will avoid this setting however, because it leads back to the perfect hedging case, in which $dY \equiv 0$.

Summing up, we have only to concern ourselves with the sign of $(\chi dt + d\mathbf{1}_{t\geq\tau})$. The output of our analysis will thus determine the constraints we must demand on $\chi(t,S,\sigma)$ in order to prevent dY to have definite sign. At this point it will be very advisable to remember the decomposition of $d\mathbf{1}_{t\geq\tau}$ stated in equation (5),

$$dY = (\Delta C + \bar{\psi}\Delta D)\left((\chi + \lambda\mathbf{1}_{t<\tau})\, dt - \lambda dG\right), \tag{14}$$

and to inspect the properties of dG:

$$dG = \begin{cases} 0 & t \geq \tau, \\ dt - \lambda^{-1} & t < \tau \leq t + dt, \\ dt & \tau > t + dt. \end{cases}$$

It is clear that for $t \geq \tau$, the change in the portfolio reduces to $dY = (\Delta C + \bar{\psi}\Delta D)\chi dt$. But once the jump has happened, there is no financial reason for having a price that differs from the Black-Scholes price corresponding to $\sigma = \sigma_b$. And this is what we will get if we set $\chi = 0$ right after the change in the volatility.

When the jump has not yet happened the differential dY reads:

$$dY = (\Delta C + \bar{\psi}\Delta D)\left(\chi dt + \lambda(dt - dG)\right),$$

with $(dt - dG) \geq 0$. Therefore we must choose $\chi < 0$ for $t < \tau$. By compiling all this, we obtain the following formula:

$$\chi(t,S,\sigma) = -\Omega(t,S)\,\mathbf{1}_{\sigma=\sigma_a} = -\Omega(t,S)\,\mathbf{1}_{t<\tau},$$

where $\Omega(t,S)$ is a strictly positive-definite bounded function depending on t and S. This function Ω may depend, in a parametric way, on σ_a and σ_b, but also on t_0 and S_0. In fact, it may also depend in general on some parameters among those that characterize the contract specifications but, and this is a crucial point, never on all of them. We must have in mind that equation (10) must hold at least for another option D, which is different from C, otherwise the market will not be complete.

We have shown the mathematical properties that χ must fulfil although we have gone into very little depth in its financial interpretation. Let us introduce function $\Psi(t,S,\sigma)$,

$$\Psi(t,S,\sigma) = (\lambda - \Omega(t,S))\;\mathbf{1}_{t<\tau},$$

in equation (14),

$$dY = (\Delta C + \bar{\psi}\Delta D)\left(\Psi dt - \lambda dG\right),$$

and then evaluate the conditional expectation of dY, for a given value of S. Since G is a martingale, $E[dG] = 0$, it is thus clear that $E[dY|S] = E[\Psi|S](\Delta C + \bar{\psi}\Delta D)dt$. Thus $\Psi(t, S, \sigma)$ measures the market price of the volatility risk, and it is exogenous to the market itself. It should be the financial agents who determine this function on the basis of their own appreciation of the actual risk. For instance, some authors [17] demand the absence of the so-call *statistical arbitrage*, i.e. $E[dY|S] = 0$. This requirement implies that $\Psi = 0$, i.e. that $\Omega = \lambda$.

5 Constant market price for volatility risk

We can now solve equation (10) under appropriate conditions. For example, we shall begin by assuming that $\Omega = \bar{\lambda}$ is constant, but not necessarily equal to λ,

$$\partial_t C + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} C - rC + rS\partial_S C + \bar{\lambda}\Delta C \mathbf{1}_{t<\tau} = 0. \quad (15)$$

This implies that the risk is considered uniform in time. We will also consider that the price of the option is constrained by the final condition:

$$C(T, S, \sigma; K, T) = \Phi(S; K, T),$$

which means that it will be a European-style option, where the price of the derivative in a fixed instant in the future, the maturity time, only depends on the actual value of the underlying at that moment and on some reference value, the strike, K. The function Φ, the payoff, will discriminate between the options within a same family. For instance, for the plain *vanilla* call we have:

$$C(T, S, \sigma; K, T) = \max(S(T) - K, 0). \quad (16)$$

In addition, the mathematical nature of equation (10) requires the solution to satisfy two extra boundary conditions which, in this case, read

$$C(t, 0, \sigma; K, T) = 0, \text{ and}, \lim_{S\to\infty} \frac{C(t, S, \sigma; K, T)}{S} = 1.$$

Obviously, we have plenty of other Φ functions, such as the Binary call where the payoff is

$$C(T, S, \sigma; K, T) = \mathbf{1}_{S(T)\geq K}, \quad (17)$$

and where the boundary conditions to be fulfilled are

$$C(t, 0, \sigma; K, T) = 0, \text{ and}, \lim_{S\to\infty} C(t, S, \sigma; K, T) = e^{-r(T-t)}.$$

We will not specify now a single function Φ, but for the moment we will treat all the suitable candidates at once. Moreover, we will use the term "Black-Scholes price", C^{BS}, as a synonymous of the solution of the Black-Scholes equation for the given payoff, without any further distinction.

This will be the case when considering equation (15) for $\tau \leq t$, since then it reduces to the Black-Scholes model:

$$\partial_t C + \frac{1}{2}\sigma_b^2 S^2 \partial^2_{SS} C - rC + rS\partial_S C = 0,$$

whose solution is accordingly

$$C(t, S, \sigma_b; K, T) = C^{\mathrm{BS}}(t, S, \sigma_b; K, T) \equiv C_b^{\mathrm{BS}}.$$

Nevertheless, we will show the main guidelines for solving it, because this will illustrate more sophisticated problems to come. The first step is to introduce two new variables, $t^* = T - t$ and $x = \log(S) + \left(r - \sigma_b^2/2\right)(T - t)$, and to assume that C depends on its own arguments only through them:

$$C(t, S, \sigma_b; K, T) = e^{-r(T-t)} V\left(T - t, \log(S) + \left(r - \frac{\sigma_b^2}{2}\right)(T - t); K\right).$$

This assumption implies the existence of a function of two variables $V(t^*, x; K)$ that obeys the following differential equation:

$$\partial_{t^*} V = \frac{1}{2}\sigma_b^2 \partial^2_{xx} V.$$

Note that t^* represents a reversion of the time arrow, which now starts at maturity. We have thus transformed our final condition into an initial one:

$$V(0, x_0; K) = \Phi(e^{x_0}; K).$$

This problem has a straightforward solution:

$$V(t^*, x; K) = \int_{-\infty}^{+\infty} dx_0 \Phi(e^{x_0}; K) \frac{1}{\sqrt{2\pi\sigma_b^2 t^*}} e^{-\frac{(x-x_0)^2}{2\sigma_b^2 t^*}}, \quad (18)$$

and therefore,

$$C(t, S, \sigma_b; K, T) = e^{-r(T-t)} \int_{-\infty}^{+\infty} \frac{dx_0 \Phi(e^{x_0}; K)}{\sqrt{2\pi\sigma_b^2 (T-t)}} e^{-\frac{\left(\log(S) + \left(r - \sigma_b^2/2\right)(T-t) - x_0\right)^2}{2\sigma_b^2 (T-t)}}, \quad (19)$$

which is the Black-Scholes price. When $\tau > t$ the equation for $C(t, S, \sigma_a; K, T)$ is a little more complex:

$$\partial_t C + \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} C - rC + rS\partial_S C + \bar{\lambda}\left(C_b^{\mathrm{BS}} - C\right) = 0.$$

The last term comes from the $\chi\Delta C$ contribution. The key point is to realize that in the expression for ΔC appears, not only $C(t, S, \sigma_b; K, T)$, which we have found in equation (19), but also $C(t, S, \sigma_a; K, T)$, the unknown quantity. The procedure to follow is very similar to the one for the previous case. We will again use variable t^*, and define ξ as $\xi = \log(S) + \left(r - \sigma_a^2/2\right)(T - t)$. In fact, x relates to ξ through $x = \xi + \left(\sigma_a^2 - \sigma_b^2\right) t^*/2$, which will be useful in a forthcoming step. Now we again assume a particular dependence on the price of this new variables,

$$C(t, S, \sigma_a; K, T) = e^{-(r+\bar{\lambda})(T-t)} Z\left(T - t, \log(S) + \left(r - \frac{\sigma_a^2}{2}\right)(T - t); K\right),$$

where $Z(t^*, \xi; K)$ obeys the following equation,

$$\partial_{t^*} Z = \frac{1}{2}\sigma_a^2 \partial_{\xi\xi}^2 Z + \bar{\lambda} e^{\bar{\lambda} t^*} V\left(t^*, \xi + \frac{\sigma_a^2 - \sigma_b^2}{2} t^*; K\right),$$

with the function V of equation (18); and the corresponding initial condition,

$$Z(0, \xi_0; K) = \Phi(e^{\xi_0}; K).$$

After some algebra, its solution reads

$$C(t, S, \sigma_a; K, T) = e^{-\bar{\lambda}(T-t)} C^{\mathrm{BS}}(t, S, \sigma_a; K, T) + \bar{\lambda} \int_t^T du\, e^{-\bar{\lambda}(u-t)} C^{\mathrm{BS}}(t, S, \bar{\sigma}(u-t, T-t); K, T),$$

where some sort of "effective variance", $\bar{\sigma}(t_a, t_b)$, has been introduced:

$$\bar{\sigma}^2(t_a, t_b) \equiv \frac{\sigma_a^2 t_a + \sigma_b^2 (t_b - t_a)}{t_b}. \quad (20)$$

Note that $\bar{\sigma}(0, T-t) = \sigma_b$ and $\bar{\sigma}(T-t, T-t) = \sigma_a$. This property can be used for compacting the solution. We can now perform a typical integration by parts inside the integral sign and recover:

$$C(t, S, \sigma_a; K, T) = C^{\mathrm{BS}}(t, S, \sigma_b; K, T) + \int_t^T du\, e^{-\bar{\lambda}(u-t)} \partial_u C^{\mathrm{BS}}(t, S, \bar{\sigma}(u-t, T-t); K, T). \quad (21)$$

The main benefit of the last operation is that equation (21) can be easily combined with equation (19), thus yielding:

$$C(t, S, \sigma; K, T) = C^{\mathrm{BS}}(t, S, \sigma_b; K, T) + \mathbf{1}_{t<\tau} \int_t^T du\, e^{-\bar{\lambda}(u-t)} \partial_u C^{\mathrm{BS}}(t, S, \bar{\sigma}(u-t, T-t); K, T). \quad (22)$$

Note that this result is not sensitive to whether τ is smaller than T, or not. It is also striking that it is $\bar{\lambda}$, not λ, which appears in the final formula. This is similar to the well-known fact that is not μ but r which counts in the valuation of the option in the Black-Scholes scheme. This latter is often interpreted *as if* the constant drift in equation (1) was in fact rdt. In Section 7 we will discuss further how the former replacement is equivalent to considering that the market acts as if the mean transition time is just $\bar{\lambda}^{-1}$.

So far, we have reproduced the framework mentioned by Herzel [14], and obtained the same formal solution to the problem when the market price of volatility risk is constant. To be strict, we have to point out that our output agrees with his expression for $t = 0$, which is, although unnumbered, the first option price given in Herzel's paper. After that, he generalizes his formulation for any later instant of time, $0 \le s \le T$. Unfortunately, there is an erratum in this extension. The limits in the definite integral in his equation (4.28) should thus be 0 and $T - s$, instead of s and T. In an equivalent way, t should be replaced by $t - s$, with the rest of the formula, including dt, unchanged.

Depending on the payoff function, the integral that appears in equation (22) can be computed, and analytic expressions for the option price can be obtained. Let us consider the two examples that appear at the beginning of the present section: the plain *vanilla* call, and the more exotic Binary call. In both cases, the explicit form of the final expression depends on the relative values of σ_a, σ_b and $\bar{\lambda}$. Thus, if $0 < \sigma_b < \sigma_a$ or $0 < \bar{\lambda} < \frac{\sigma_b^2 - \sigma_a^2}{8}$, the price of the classical European call reads:

$$\begin{aligned} C(t, S, \sigma; K, T)/S = {} & \mathcal{N}\left(\frac{\alpha}{\sigma_b} + \sigma_b \beta\right) - e^{-2\alpha\beta} \mathcal{N}\left(\frac{\alpha}{\sigma_b} - \sigma_b \beta\right) \\ & + \mathbf{1}_{t<\tau} \frac{\beta}{\gamma} \exp\left(\frac{1}{2}(\gamma^2 - \beta^2)\sigma_b^2 - \alpha\beta\right) \\ & \times \left\{ e^{\alpha\gamma} \left[\mathcal{N}\left(\frac{\alpha}{\sigma_a} + \sigma_a \gamma\right) - \mathcal{N}\left(\frac{\alpha}{\sigma_b} + \sigma_b \gamma\right)\right] \right. \\ & \left. - e^{-\alpha\gamma} \left[\mathcal{N}\left(\frac{\alpha}{\sigma_a} - \sigma_a \gamma\right) - \mathcal{N}\left(\frac{\alpha}{\sigma_b} - \sigma_b \gamma\right)\right] \right\}, \end{aligned} \quad (23)$$

whereas if $0 < \frac{\sigma_b^2 - \sigma_a^2}{8} < \bar{\lambda}$, the solution, although real, involves the use of complex calculus:

$$\begin{aligned} C(t, S, \sigma; K, T)/S = {} & \mathcal{N}\left(\frac{\alpha}{\sigma_b} + \sigma_b \beta\right) - e^{-2\alpha\beta} \mathcal{N}\left(\frac{\alpha}{\sigma_b} - \sigma_b \beta\right) \\ & + \mathbf{1}_{t<\tau} \frac{2\beta}{\bar{\gamma}} \exp\left(-\frac{1}{2}(\bar{\gamma}^2 + \beta^2)\sigma_b^2 - \alpha\beta\right) \\ & \times \Im \left\{ e^{i\alpha\bar{\gamma}} \left[\mathcal{N}\left(\frac{\alpha}{\sigma_a} + i\sigma_a \bar{\gamma}\right) - \mathcal{N}\left(\frac{\alpha}{\sigma_b} + i\sigma_b \bar{\gamma}\right)\right] \right\}. \end{aligned} \quad (24)$$

In writing these results, we have shuffled the free variables and parameters in order to keep the expressions as readable as possible, only defining four new quantities:

$$\begin{aligned} \alpha &= \frac{\log(S/K) + r(T-t)}{\sqrt{T-t}}, \\ \beta &= \frac{1}{2}\sqrt{T-t}, \\ \gamma &= \sqrt{\frac{2\bar{\lambda}(T-t)}{\sigma_a^2 - \sigma_b^2} + \beta^2}, \text{ and} \\ \bar{\gamma} &= \sqrt{\frac{2\bar{\lambda}(T-t)}{\sigma_b^2 - \sigma_a^2} - \beta^2}. \end{aligned}$$

We have also introduced the cumulative distribution function for a Normal probability density:

$$\mathcal{N}(\cdot) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\cdot} e^{-u^2/2} du,$$

or its natural extension to the complex plane when needed.

The border-line case $\bar{\lambda} = \frac{\sigma_b^2 - \sigma_a^2}{8}$, can be obtained from either of the two previous expressions for $C(t, S, \sigma; K, T)$, because when $t \neq \tau$ the price of the option is continuous in all variables and parameters. We must only set $\gamma = 0$ or $\bar{\gamma} = 0$, depending on the starting point:

$$C(t, S, \sigma; K, T)/S = \mathcal{N}\left(\frac{\alpha}{\sigma_b} + \sigma_b \beta\right) - e^{-2\alpha\beta} \mathcal{N}\left(\frac{\alpha}{\sigma_b} - \sigma_b \beta\right) + \mathbf{1}_{t<\tau} 2\beta \exp\left(-\frac{1}{2}\beta^2 \sigma_b^2\right) \times \left\{ \frac{\sigma_a e^{-\frac{\alpha^2}{2\sigma_a^2}} - \sigma_b e^{-\frac{\alpha^2}{2\sigma_b^2}}}{\sqrt{2\pi}} + \alpha \left[\mathcal{N}\left(\frac{\alpha}{\sigma_a}\right) - \mathcal{N}\left(\frac{\alpha}{\sigma_b}\right) \right] \right\}. \tag{25}$$

In the special case that the *discounted moneyness* is equal to one, i.e. $S = Ke^{-r(T-t)}$ or $\alpha = 0$, equation (25) reduces to a very simple expression:

$$C(t, S, \sigma; Se^{r(T-t)}, T)/S = 2\mathcal{N}(\sigma_b \beta) - 1 + \mathbf{1}_{t<\tau} 2\beta \exp\left(-\frac{1}{2}\beta^2 \sigma_b^2\right) \times \left\{ \frac{\sigma_a - \sigma_b}{\sqrt{2\pi}} \right\}.$$

The results for a Binary or Digital call are very similar to the preceding ones in overall terms. For the first range, $0 < \sigma_b < \sigma_a$ or $0 < \bar{\lambda} < \frac{\sigma_b^2 - \sigma_a^2}{8}$,

$$C(t, S, \sigma; K, T) e^{r(T-t)} = \mathcal{N}\left(\frac{\alpha}{\sigma_b} - \sigma_b \beta\right) + \mathbf{1}_{t<\tau} \frac{1}{2} \exp\left(\frac{1}{2}(\gamma^2 - \beta^2)\sigma_b^2 + \alpha\beta\right) \times \left\{ \left(1 - \frac{\beta}{\gamma}\right) e^{\alpha\gamma} \left[\mathcal{N}\left(\frac{\alpha}{\sigma_a} + \sigma_a \gamma\right) - \mathcal{N}\left(\frac{\alpha}{\sigma_b} + \sigma_b \gamma\right) \right] + \left(1 + \frac{\beta}{\gamma}\right) e^{-\alpha\gamma} \left[\mathcal{N}\left(\frac{\alpha}{\sigma_a} - \sigma_a \gamma\right) - \mathcal{N}\left(\frac{\alpha}{\sigma_b} - \sigma_b \gamma\right) \right] \right\}. \tag{26}$$

while, in the complementary case $0 < \frac{\sigma_b^2 - \sigma_a^2}{8} < \bar{\lambda}$, the price of the option is:

$$C(t, S, \sigma; K, T) e^{r(T-t)} = \mathcal{N}\left(\frac{\alpha}{\sigma_b} - \sigma_b \beta\right) + \mathbf{1}_{t<\tau} \exp\left(-\frac{1}{2}(\bar{\gamma}^2 + \beta^2)\sigma_b^2 + \alpha\beta\right) \times \Re\left\{ \frac{\bar{\gamma} + i\beta}{\bar{\gamma}} e^{i\alpha\bar{\gamma}} \left[\mathcal{N}\left(\frac{\alpha}{\sigma_a} + i\sigma_a \bar{\gamma}\right) - \mathcal{N}\left(\frac{\alpha}{\sigma_b} + i\sigma_b \bar{\gamma}\right) \right] \right\}. \tag{27}$$

The limit case $\bar{\lambda} = \frac{\sigma_b^2 - \sigma_a^2}{8}$ can easily be reproduced from the expressions above. Note that the same definitions of α, β, γ and $\bar{\gamma}$ apply.

In Section 7 we will present some numerical examples of the explicit solutions we have introduced here.

6 New market prices of risk

In this section, we will consider some non-constant candidates for $\Omega(t, S)$. The main challenge is choosing suitable functions, not only from the mathematical point of view, but also with a clear financial interpretation. They should represent a credible reaction of the market to the risk associated with the possible change in volatility. Otherwise, the new results will be merely an exercise in applied mathematics.

For instance, it is easy to notice that a wide range of new solutions can be obtained following an approach very similar to the development carried out in the previous section, with little extra effort. Let us consider the case of a Ω depending on all the time magnitudes involved:

$$\Omega = \eta(t; t_0, T),$$

with $\eta(t; t_0, T) > 0$. The solution for $t = t_0$ is simply[2]:

$$C(t_0, S_0, \sigma; K, T) = C^{\text{BS}}(t_0, S_0, \sigma_b; K, T) + \mathbf{1}_{t_0<\tau} \int_{t_0}^{T} du\, e^{-\int_{t_0}^{u} dt' \eta(t'; t_0, T)} \times \partial_u C^{\text{BS}}(t_0, S_0, \bar{\sigma}(u - t_0, T - t_0); K, T). \tag{28}$$

With this formal expression we can generate an infinite set of valid prices, depending on the choice of an arbitrary function. With the same spirit as in the preceding section, we could say that the market behaves *as if* τ were replaced by another random variable $\bar{\tau}$, governed by the alternative law:

$$P(t_0 < \bar{\tau} \leq t) = 1 - e^{-\int_{t_0}^{t} dt' \eta(t'; t_0, T)}.$$

However, it should be recalled that this change does not affect the transition time itself, which depends on τ, but the market's perception of the probability of the change taking place. We will consider this interpretation in more depth later.

A plausible requirement that may help us to discard candidates is to demand that the final solution depends on $T - t_0$. This means that $\eta(t; t_0, T) = f(t - t_0; T - t_0)$, which is a rather soft constraint. In Appendix A we will consider a case in which an investor tries to reproduce the future behaviour of options using only shares and bonds. Since there is no traded asset other than options that quote the volatility, this effort will be in vain. However, the option price obtained:

$$\bar{C}(t_0, S_0, \sigma_0; K, T) = C^{\text{BS}}(t_0, S_0, \sigma_b; K, T) + \mathbf{1}_{t_0<\tau} \int_{t_0}^{T} du \frac{1 + e^{-\lambda(u - t_0)}}{2} \times \partial_u C^{\text{BS}}(t_0, S_0, \bar{\sigma}(u - t_0, T - t_0); K, T), \tag{29}$$

[2] Given that, up to this point, the notation did not lead to misunderstanding, we have not stressed the difference between t_0, the actual time in which the options is evaluated, and t, a generic instant of time, $t_0 \leq t \leq T$.

is still valid. It is straightforward to check that if we replace

$$\eta(t;t_0,T)=\eta(t-t_0)=\lambda\frac{e^{-\lambda(t-t_0)}}{1+e^{-\lambda(t-t_0)}}, \tag{30}$$

in equation (28), we will recover equation (29). Furthermore, we can rewrite this expression in terms of the Black-Scholes price for $\sigma=\sigma_a$, and $C^\lambda(t_0,S_0,\sigma_0;K,T)$, the solution found in the previous section for the constant case $\bar{\lambda}=\lambda$:

$$\bar{C}(t_0,S_0,\sigma_0;K,T)=\frac{1}{2}\left[C^{\rm BS}(t_0,S_0,\sigma_a;K,T) + C^\lambda(t_0,S_0,\sigma_0;K,T)\right]. \tag{31}$$

This expression may help us to reinterpret the result from another point of view. The first term would be the fair price if the volatility did not change after the jump, whereas the second one is possibly the most neutral choice for pricing the new risk. This investor thus acts *as if* he was skeptic about the correctness of the model, *as if* he really thought that when the transition takes place (e.g. when the expected news are published) σ may *or may not* change. And he weights the two scenarios in the same way.

Clearly, we can obtain solutions for the entire range of confidence levels of the accuracy of the model. Thus, if we set q as equal to the "likelihood" that the volatility actually varies when the jump takes place, the function η that expresses this risk evaluation is

$$\eta(t;t_0,T)=\eta(t-t_0)=\lambda\frac{e^{-\lambda(t-t_0)}}{(1-q)/q+e^{-\lambda(t-t_0)}}.$$

The only constraint is that we cannot neglect all the risk by merely setting $q=0$. Therefore any $0<q\le 1$ leads to a valid price,

$$\bar{C}^{(q)}(t_0,S_0,\sigma_0;K,T)=(1-q)C^{\rm BS}(t_0,S_0,\sigma_a;K,T) + qC^\lambda(t_0,S_0,\sigma_0;K,T),$$

although in the absence of further information $q=1/2$ and $q=1$ seem to be the only privileged values. In Section 7 we will explore the consequences of this reluctance by the investor to accept equation (2) as a valid description of the future behaviour of the volatility.

We will end this section by stating another possible scenario. We will focus on the case that the volatility risk premium is determined by the price of the stock, without any explicit temporal evolution prior to the jump. We may thus assume that Ω depends on the two currency-related magnitudes, the variable S and the parameter K:

$$\Omega=\Lambda(S;K),$$

with $\Lambda(S;K)>0$. Since the market price of the volatility risk is not a function of T, we can still fully hedge the changes in the price of a given option using other options with different expiration time as well as shares and bonds. The presence of the striking price in the risk premium brings an interesting new dimension to the problem. Let us analyse a simple example in which:

$$\Lambda(S;K)=\lambda_a\mathbf{1}_{S<K}+\lambda_b\mathbf{1}_{S\ge K}, \tag{32}$$

with $\lambda_a,\lambda_b>0$. Such a choice seems to have its natural origin in a market in which the practitioners think that it is more or less probable that the expected event will take place depending on the fact that the stock is out of the money, i.e. $S<K$, or not.

In order to show that this interpretation has some financial meaning, we will present a fictitious practical situation. Let us imagine, for instance, that a firm plans to take over another one in the near future, but a time that is not yet apparent. The reason for expecting a higher volatility of the security after the purchase can be easily understood if the resulting company is more exposed to economical or political fluctuations. This could be a canonical situation for applying a constant volatility risk premium. Let us assume nonetheless that the firm has sold a large amount of calls in the past with the same striking price. If the price of the corporate shares exceeds the level defined by the strike of these options at maturity, the company will need some reserve of funds for covering the demands of the option holders. The market could therefore consider that the takeover might be delayed within this frame.

We have decided to study the case in which λ_a and λ_b are constant, instead of any other smoother transition (which may have a fit better in the example given) for two main reasons. On the one hand it will be easier to analyse the recently introduced modification, in terms of the previous results. A naive reasoning leads to the conclusion that the option price in this circumstances must fulfil $C(t_0,S_0,\sigma_0;K,T)\sim C^{\lambda_a}(t_0,S_0,\sigma_0;K,T)$ if $S_0\ll K$, and $C(t_0,S_0,\sigma_0;K,T)\sim C^{\lambda_b}(t_0,S_0,\sigma_0;K,T)$ when $S_0\gg K$. On the other hand, it appears at first glance that we could adapt the procedures above in order to solve this problem. Unfortunately that is not the case as far as we know. The fact is that we have had to resort to numerical computation for obtaining the results presented below.

7 Numerical analysis

In this section we will analyse and compare the different solutions we have found, and eventually represent some of them. Since all the possible candidates to be the fair price collapse to the Black-Scholes solution if the jump takes place, we will implicitly assume that $t_0<\tau$ from now on.

We will follow the guidelines of the previous exposition and thus we will begin dealing with constant volatility risk premium. Firstly, we will show the example of a typical European call. Remember that in this case we have closed expressions —equation (23) or equation (24)— for evaluating the option price. However, for the sake of completeness, we will also consider the output of numerical methods based on statistical formulas. Within this approach we must express the option price in the form of an expected value of the discounted payoff, under some appropriate probability density function:

$$C(t_0, S_0, \sigma_0; K, T) = E^Q[e^{-r(T-t_0)}\Phi(S(T); K)]. \quad (33)$$

If we want $C(t_0, S_0, \sigma_0; K, T) = C^{\bar{\lambda}}(t_0, S_0, \sigma_0; K, T)$ the final asset value $S(T)$ in equation (33) must be expressed in the following terms

$$S(T) = S_0 e^{\left(r-\sigma_a^2/2\right)(T-t_0)+\sigma_a \overline{W}(T-t_0)} \mathbf{1}_{T<\bar{\tau}} + S(\bar{\tau}) e^{\left(r-\sigma_b^2/2\right)(T-\bar{\tau})+\sigma_b\left[\overline{W}(T-t_0)-\overline{W}(\bar{\tau}-t_0)\right]} \mathbf{1}_{T\geq\bar{\tau}},$$

with

$$S(\bar{\tau}) = S_0 e^{\left(r-\sigma_a^2/2\right)(\bar{\tau}-t_0)+\sigma_a \overline{W}(\bar{\tau}-t_0)}.$$

Thus, under the equivalent measure Q, μ shall be replaced by r, as usual, and a new Brownian motion with zero mean and variance equal to $t-t_0$, must be introduced: $\overline{W}(t-t_0)$. The random variable $\bar{\tau}$ follows also an exponential law:

$$P(t_0 < \bar{\tau} \leq t) = 1 - e^{-\bar{\lambda}(t-t_0)}.$$

As we have already argued, when $\bar{\lambda} \neq \lambda$ we can consider in practice that the model is not accurate in forecasting the actual mean transition time, and that it should be replaced by another value. This set-up is suitable for performing Monte Carlo simulations, and comparing them with the exact results. Since it is not our intention to add superfluous complexity we will concentrate in the case that the original λ (and τ) is used. In Figure 1 we therefore find, the numerical estimation of the *vanilla* call price together with the graphical representation of equation (24) for the parameter set reported there, showing an excellent agreement.

Also in Figure 1 we present the results associated with another criterion for the volatility risk premium: the time-dependent function that leads to equation (31). In this case, if we model the share price at maturity according to

$$S(T) = S_0 e^{\left(r-\sigma_a^2/2\right)(T-t_0)+\sigma_a \overline{W}(T-t_0)} \mathbf{1}_{T<\tau} + \frac{1}{2} S(\tau) e^{\left(r-\sigma_a^2/2\right)(T-\tau)+\sigma_a\left[\overline{W}(T-t_0)-\overline{W}(\tau-t_0)\right]} \mathbf{1}_{T\geq\tau} + \frac{1}{2} S(\tau) e^{\left(r-\sigma_b^2/2\right)(T-\tau)+\sigma_b\left[\overline{W}(T-t_0)-\overline{W}(\tau-t_0)\right]} \mathbf{1}_{T\geq\tau},$$

with the same specifications for τ, $\overline{W}(t-t_0)$ and $S(\tau)$ as in the past scenario, we can use equation (33) to recover $\bar{C}(t_0, S_0, \sigma_0; K, T)$. Note once again that this particular choice for the market assessment of risk leads, when $\tau < T$, to an expression for $S(T)$ that is the arithmetic mean of the two possible paths: one in which the volatility is σ_b right after the jump, and the other that considers that σ_a remains unchanged. This is why we have stressed the necessity of dissociating the existence of a distinctive time value τ, and the innovation that it carries. We can be sure that forthcoming news may affect the market and, at the same time, can only guess about its final effect. Thus equation (30) leads to a more conservative risk analysis, in the sense that this price is nearer the Black-Scholes value corresponding to σ_a, whereas $C^{\lambda}(t_0, S_0, \sigma_0; K, T)$ anticipates the future change in the volatility more intensely. This explains the behaviour of the different call prices observed in Figure 1. Looking at Figures 1a and b, we can check that the two prices converge to the *no-jump* solution as the maturity horizon comes closer. In all instances the simulated prices fit the theoretical curves very well.

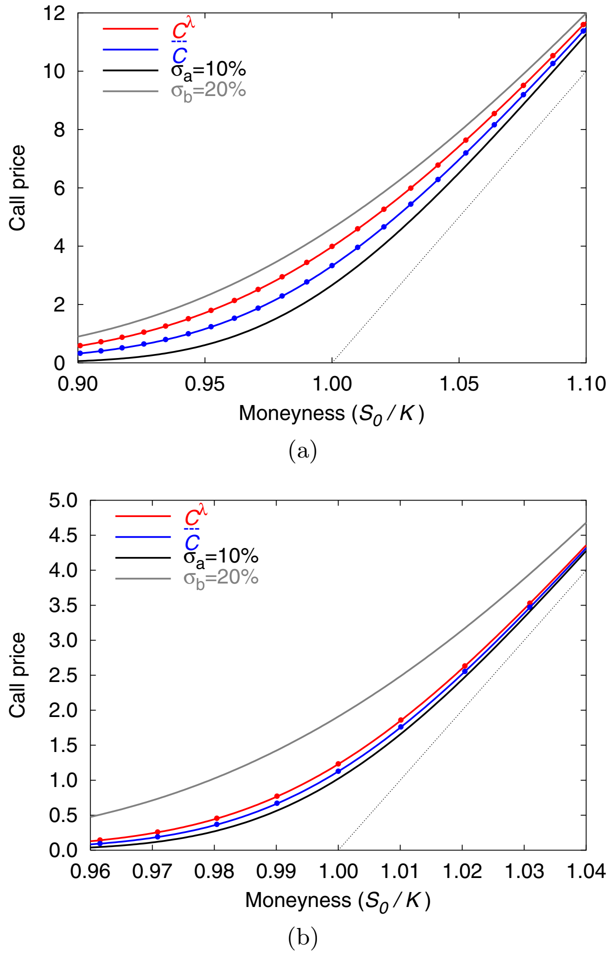
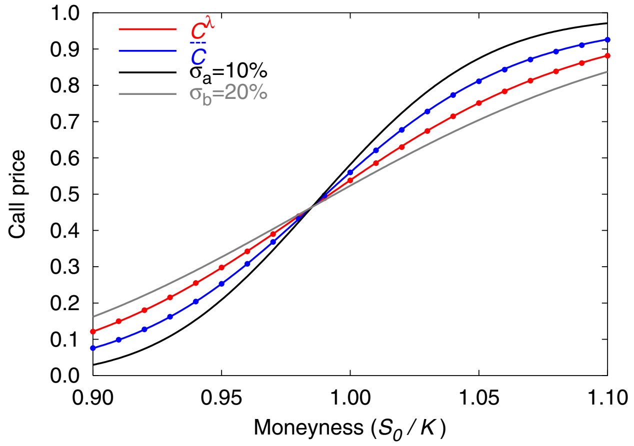
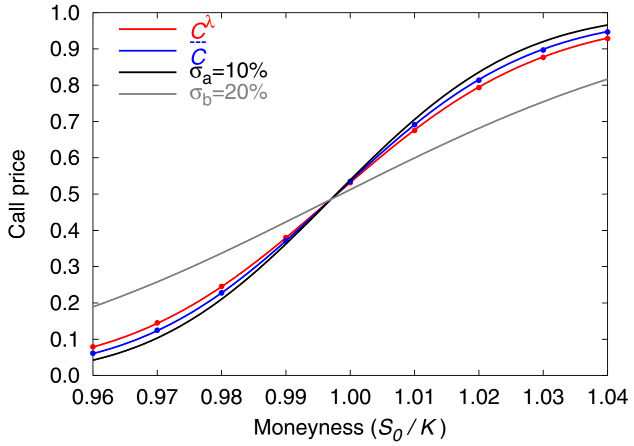


Fig. 1. Option pricing for European calls. In this figure we present the results corresponding to the payoff $\Phi(S; K, T) = \max(S(T) - K, 0)$, in terms of the initial moneyness S_0/K. We plot curves for two different maturities: (a) $T - t_0 = 0.25$ years, and (b) $T - t_0 = 0.05$ years. The numerical value of the other parameters are $r = 5\%$, $\sigma_a = 10\%$, $\sigma_b = 20\%$, $\lambda^{-1} = 0.1$ years, and $K = 100$, in suitable currency units. The red line corresponds to $C^{\lambda}(t_0, S_0, \sigma_0; K, T)$ while the blue line was obtained using $\bar{C}(t_0, S_0, \sigma_0; K, T)$. In the main text we argue that this prescription for pricing the volatility risk can be explained as a lack of confidence in the actual change in volatility after the expected event. Note that the first price is more similar to the plain Black-Scholes price with $\sigma = \sigma_b$, and that conversely the second method leads to a price closer to the Black-Scholes price for $\sigma = \sigma_a$. The discrepancy is reduced as the maturity time approaches. The solid lines were computed on the basis of exact expressions, while the dots were obtained by using an alternative Monte Carlo procedure, averaging over 100 000 replicas.

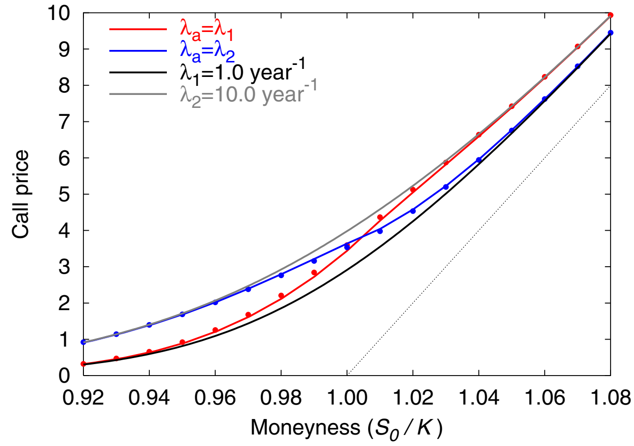


(a)

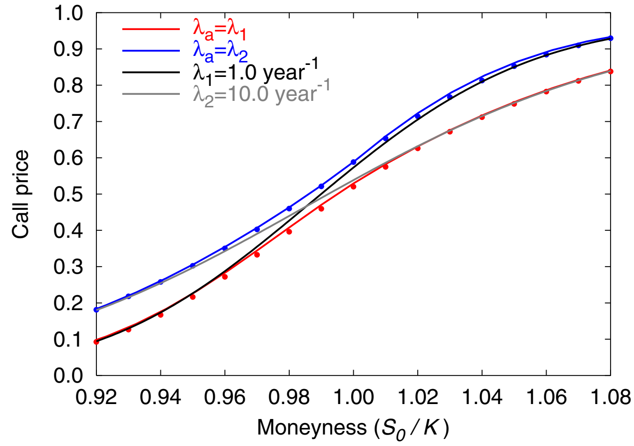


(b)

Fig. 2. Option pricing for Binary calls. In this figure we present, in terms of the initial moneyness S_0/K, the results corresponding to the payoff $\Phi(S;K,T) = \mathbf{1}_{S(T)\geq K}$. We again plot curves for two different maturities: (a) $T - t_0 = 0.25$ years, and (b) $T - t_0 = 0.05$ years. The numerical value of the other parameters are the same as in Figure 1. We also repeat the colour codes, and present both exact results (solid lines) and numerical approximations (dots), computed with the same methods and technical specifications.



(a)



(b)

Fig. 3. Option pricing for stock-dependent volatility risk valuation. This figure shows some results within a market where the price for the volatility risk switches between two levels depending whether the stock is over the strike or not: $\Lambda(S;K) = \lambda_a \mathbf{1}_{S<K} + \lambda_b \mathbf{1}_{S\geq K}$. We represent the price of the option, in terms of the initial moneyness S_0/K, for two payoffs: (a) European call, and (b) Binary call, when maturity is $T - t_0 = 0.25$ years. The (dark) *reference* lines correspond to $C^\lambda(t_0, S_0, \sigma_0; K, T)$ in two very different scenarios. In the first one ($\lambda_1^{-1} = 1.0$ years) it is quite unlikely that the jump will happen, whereas in the second one ($\lambda_2^{-1} = 0.1$ years) the turnabout is more conceivable. Solid lines are the result of a numerical integration of the partial differential equations while the dots were computed using Monte Carlo methods. The other parameters are the same as in the previous figures.

In Figure 2 we illustrate the case of a Binary call, under the two previous volatility risk appreciations: the constant premium that leads to equations (26) and (27), and the more cautious strategy represented by $\bar{C}(t_0, S_0, \sigma_0; K, T)$. The main features remain unchanged with just one peculiarity: all prices seem to converge for a given value of the moneyness. Unfortunately, this is just an optical effect. The two Black-Scholes prices coincide when $\alpha = \sigma_a \sigma_b \beta$. The other crossing points are within a narrow interval that includes this value, but do not fully match up.

The stock-dependent evolution of the volatility risk measure is introduced in Figure 3. In the case analysed — equation (32) with constant values for λ_a and λ_b— we have not obtained formal expressions of any kind for the price of the option. We do not even have an exact formula for $S(T)$ in equation (33) which we can take as a starting point of a Monte Carlo estimation. We must therefore rely on biased numerical methods. The approach we have taken for solving the problem has in all cases implied the discretization of the time interval: $t_0, t_1, \cdots, t_{N-1}, t_N = T$. In practice we have set $N = 250$, and used gaps with the same length. The total number of replicas used within the Monte Carlo framework was kept within $100,000$. Each value of $S(T)$

was computed following the iterative procedure:

$$S(t_{i+1}) = S(t_i)e^{\left(r-\sigma_a^2/2\right)(t_{i+1}-t_i)+\sigma_a\overline{W}(t_{i+1}-t_i)}\mathbf{1}_{t_{i+1}<\bar{\tau}} + S(t_i)e^{\left(r-\sigma_b^2/2\right)(t_{i+1}-t_i)+\sigma_b\overline{W}(t_{i+1}-t_i)}\mathbf{1}_{t_{i+1}\geq\bar{\tau}},$$

where $\overline{W}(t_{i+1}-t_i)$ is a Brownian motion with zero mean and variance equal to $t_{i+1}-t_i$, and the random variable $\bar{\tau}$ fulfills the conditional law:

$$P(\bar{\tau} \leq t_{i+1}|t_i < \bar{\tau}) = 1 - e^{-\Lambda(S(t_i);K)(t_{i+1}-t_i)}.$$

The partial differential equations were numerically figured out backward in time on a exponential grid of 100, 001 points, centred around K. The *initial* conditions were the contingent claims in (16) and (17), with their corresponding boundary conditions. The *vanilla* call price was obtained using a typical *fully explicit* FTCS procedure: a symmetrical finite-difference approximation. The Binary call was a little more delicate, and the use of a *Crank-Nicholson* scheme was necessary in the first 175 time steps.

However, the main fact is that both numerical methods (FTCS and Monte Carlo) lead eventually to very similar results. The behaviour of these solutions is as we predicted at the end of Section 6: the price gradually attains C^{λ_a} (C^{λ_b}) when the asset is below (above) the strike.

8 Conclusions

We have revisited the framework stated by Herzel, in which the dynamics of one asset S is driven by a log-normal diffusion equation with a stochastic volatility parameter σ. The volatility of this stock may jump at a random time τ from a fixed initial value σ_a to another fixed final value σ_b. No more than one such jumps is allowed. This event can model the future publication of crucial information related to this specific market, for instance.

We have introduced a procedure for obtaining fair option prices, different from that used by Herzel in his original manuscript. There, the author intensively exploits probability arguments for finding the necessary and sufficient conditions that the model must fulfil to be complete and arbitrage-free. He thus derives the equivalent martingale measure. We have employed another technique broadly used in this field. We have determined the partial differential equations that, according to Itô's formula, the option price must fulfil. We have shown that the use of a *secondary option* completes the market. After that, we have required the market to have no arbitrage and we have found the exogenous function that measures the market price of the volatility risk. We have explored the output for several choices of this function and, incidentally, we have amended some of the results presented in the original reference, where the risk premium was null.

In fact, one of the biggest benefits of our approach is when considering more sophisticated prescriptions for the market price of volatility risk. We have not only obtained closed formulas in such a cases, but we have also been able to interpret their financial meaning. We have seen how a choice for the volatility risk price can be translated into a lack of confidence in the model premises. For instance, a constant risk price other than zero plays the same role of a redefinition in the mean transition time of the jump process.

In particular, we have studied a solution in some detail that can be understood as the response of a suspicious investor, who admits the possibility that volatility will stay on the same level, although the jump (that is, the announcement) has taken place. We have also presented another scenario in which the volatility risk premium is a function of the value of the underlying asset. The final picture is equivalent to a market where the possibility that the transition takes place is felt to be more or less likely depending on the present price of the shares. Some plots with actual examples complete the exposition.

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Appendix A

We have seen that the process for valuing the option relies on the investor's opportunity to replicate the variations of the call price with an alternative portfolio. The buyer of this equivalent portfolio must nonetheless hold *secondary* options, options with the same underlying but different contract specifications. This is quite a circular situation. Let us assume for a moment that an investor wants to use only shares and bonds in order to hedge the option. It is obvious that he will fail in this task by definition, because this is only feasible if the jump time is deterministic. But in this case, equation (3) and all the derived expressions were no longer correct. We should simply consider a log-normal model with time-dependent volatility instead. In this Appendix we present an alternative approach that tries to remove the risk associated with the volatility change in a portfolio without secondary calls. We will still assume that τ is a stochastic magnitude, we will obtain results conditioned to the value of this variable, and we will finally take expectations. It will be a heuristic procedure in the sense that, in general, such a practice may lead to an incorrect option price. In the main body of the article we will check that this is not the case here.

We will therefore start by recovering equation (13) and directly setting $\bar{\psi} = 0$:

$$dY = \Delta C\left(\chi dt + d\mathbf{1}_{t\geq\tau}\right).$$

We now require that $dY = 0$. This constraint leads to:

$$\chi = -\frac{d}{dt}\mathbf{1}_{t\geq\tau} = -\delta(t-\tau),$$

where the Dirac's Delta $\delta(\cdot)$ has been used. The corresponding main equation is then:

$$\partial_t C+\frac{1}{2}\sigma^2 S^2\partial^2_{SS}C-rC+rS\partial_S C+\Delta C\delta(t-\tau)=0. \quad (A.1)$$

We will search for a solution taking the form

$$C(t,S,\sigma;K,T)= e^{-r(T-t)}U\left(T-t,\log(S)+\left(r-\frac{\sigma^2}{2}\right)(T-t);K\right),$$

based upon the two variable function $U(t^*,x;K)$, which must fulfil the following partial equation:

$$\partial_{t^*}U-\frac{1}{2}\sigma^2\partial^2_{xx}U=\Delta U\delta(T-\tau-t^*).$$

We therefore consider the Fourier-Laplace transform of $U(t^*,x;K)$,

$$\widehat{U}(s,\omega;K)=\int_0^{+\infty}dt^* e^{-st^*}\int_{-\infty}^{+\infty}dx\, e^{i\omega x}\,U(t^*,x;K),$$

that follows the simpler equation:

$$s\widehat{U}-\widetilde{U}_0+\frac{1}{2}\sigma^2\omega^2\widehat{U}-\Delta\widetilde{U}_{T-\tau}e^{-s(T-\tau)}\mathbf{1}_{\tau\le T}=0,$$

where the tilde stands for the Fourier transform of the corresponding object. Thus $\widetilde{U}_0(\omega;K)\equiv\widetilde{U}(t^*=0,\omega,\sigma;K)$, and it does not depend on σ. On the other hand, $\Delta\widetilde{U}_{T-\tau}\equiv\widetilde{U}(T-\tau,\omega,\sigma_b;K)-\widetilde{U}(T-\tau,\omega,\sigma_a;K)$. Now we can isolate all the explicit dependence on the Laplace variable s,

$$\widehat{U}(s,\omega,\sigma;K)=\frac{1}{s+\sigma^2\omega^2/2}\times\left\{\widetilde{U}_0+\Delta\widetilde{U}_{T-\tau}e^{-s(T-\tau)}\mathbf{1}_{\tau\le T}\right\},$$

and perform an inverse transformation,

$$\widetilde{U}(t^*,\omega,\sigma;K)=\widetilde{U}_0 e^{-\sigma^2\omega^2 t^*/2}+\Delta\widetilde{U}_{T-\tau}e^{-\sigma^2\omega^2(t^*-T+\tau)/2}\mathbf{1}_{T-t^*<\tau\le T}. \quad (A.2)$$

Obviously, the second term only makes a contribution when the jump is made between t, and the maturity, T. When $t^*\le T-\tau$, i.e. $\tau\le t$ and $\sigma=\sigma_b$, equation (A.2) reduces to,

$$\widetilde{U}(t^*,\omega,\sigma_b;K)=\widetilde{U}_0 e^{-\sigma_b^2\omega^2 t^*/2}=\widetilde{U}^{\mathrm{BS}}(t^*,\omega,\sigma_b;K),$$

which leads to

$$C(t,S,\sigma_b;K,T)=C^{\mathrm{BS}}(t,S,\sigma_b;K,T),$$

a riskless price.

When $t^*>T-\tau$, i.e., when $t<\tau$ and $\sigma=\sigma_a$, but $\tau>T$, the main equation also takes a simple form,

$$\widetilde{U}(t^*,\omega,\sigma_a;K)=\widetilde{U}_0 e^{-\sigma_a^2\omega^2 t^*/2}=\widetilde{U}^{\mathrm{BS}}(t^*,\omega,\sigma_a;K).$$

In this case, since the change in volatility occurs *after* the maturity of the contract, the price reduces to a plain Black-Scholes model without any jump in volatility,

$$C(t,S,\sigma_a;K,T|T<\tau)=C^{\mathrm{BS}}(t,S,\sigma_a;K,T).$$

This scenario thus again has no risk associated with it.

Finally, when $t^*>T-\tau$ and $\tau\le T$, all the terms contribute to a more complex expression,

$$\widetilde{U}(t^*,\omega,\sigma_a;K)=\widetilde{U}_0 e^{-\sigma_a^2\omega^2 t^*/2}+\Delta\widetilde{U}_{T-\tau}e^{-\sigma_a^2\omega^2(t^*-T+\tau)/2}. \quad (A.3)$$

It should be recalled that $\widetilde{U}_{T-\tau}$ is a term that counts *only* for the variation in $\widetilde{U}$ due to the change in the volatility, when it takes place. Thus $\widetilde{U}(T-\tau,\omega,\sigma_b;K)=\widetilde{U}^{\mathrm{BS}}(T-\tau,\omega,\sigma_b;K)$. The other term can be obtained by self-consistency. We will start from equation (A.3) and take a limit:

$$\widetilde{U}(T-\tau,\omega,\sigma_a;K)=\lim_{t^*\to T-\tau}\widetilde{U}(t^*,\omega,\sigma_a;K)= \lim_{t^*\to T-\tau}\widetilde{U}_0 e^{-\sigma_a^2\omega^2 t^*/2}+\Delta\widetilde{U}_{T-\tau}e^{-\sigma_a^2\omega^2(t^*-T+\tau)/2},$$

which leads to

$$\widetilde{U}(T-\tau,\omega,\sigma_a;K)= \frac{1}{2}\left[\widetilde{U}^{\mathrm{BS}}(T-\tau,\omega,\sigma_a;K)+\widetilde{U}^{\mathrm{BS}}(T-\tau,\omega,\sigma_b;K)\right].$$

Now we will introduce this result back into equation (A.3), and obtain

$$\widetilde{U}(t^*,\omega,\sigma_a;K)= \frac{1}{2}\widetilde{U}_0\left[e^{-\sigma_a^2\omega^2 t^*/2}+e^{-\bar{\sigma}^2(t^*-T+\tau,t^*)\omega^2 t^*/2}\right],$$

where $\bar{\sigma}(t_a,t_b)$ is the same that in equation (20). Therefore

$$C(t,S,\sigma_a;K,T|t<\tau\le T)= \frac{1}{2}\left[C^{\mathrm{BS}}(t,S,\sigma_a;K,T)+C^{\mathrm{BS}}(t,S,\bar{\sigma}(\tau-t,T-t);K,T)\right].$$

Finally, in order to obtain an expression for $t<\tau$ that does not depend on future information, we will compute the expected value of the previous conditioned solutions:

$$\begin{aligned}\bar{C}(t,S,\sigma_a;K,T)&=E\left[C(t,S,\sigma_a;K,T|\tau=u)\right]\\&=\frac{\lambda}{2}\int_t^T du\left[C^{\mathrm{BS}}(t,S,\sigma_a;K,T)\right.\\&\quad\left.+\,C^{\mathrm{BS}}(t,S,\bar{\sigma}(u-t,T-t);K,T)\right]e^{-\lambda(u-t)}\\&\quad+\lambda\int_T^{+\infty}du\,C^{\mathrm{BS}}(t,S,\sigma_a;K,T)e^{-\lambda(u-t)},\end{aligned}$$

an expression that reduces to

$$\bar{C}(t,S,\sigma_a;K,T)=C^{\mathrm{BS}}(t,S,\sigma_b;K,T)+\int_t^T du\,\frac{1+e^{-\lambda(u-t)}}{2}\partial_u C^{\mathrm{BS}}(t,S,\bar{\sigma}(u-t,T-t);K,T).$$

a formula that does not fulfil equation (A.1), but which is still a valid solution.

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